APPROXIMATE SOLUTIONS TO NON-LINEAR DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORM TECHNIQUES

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THESIS

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APPROXIMATE SOLUTIONS

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USING LAPLACE TRANSFORM TECHNIQUES

by

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ABSTRACT

At the present time the primary method of obtaining solutions to non-linear differential equations is by means of the digital computer and numerical techniques. A method is here proposed to find an approximate mathematical expression through the use of Laplace Transform techniques. Thus, the Laplace Transform concept is extended to the solution of non-linear differential equations.

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CHAPTER 1

1.0 Introduction

Laplace Transforms have long been employed in the solution of linear differential equations. Baycura [ref. 1] proposed a method of obtaining an approximate solution to a non-linear differential equation by using a linearized transform to obtain the approximation. In a later paper [ref. 2] it was proposed that an exact solution might be found by developing an expression for the Laplace Transform of a non-linear term. The result was that the solution was not exact, but very similar in form.

These results are examined in detail here, and a general expression for the non-linear transform is developed. Further, the concept is extended to non-homogeneous equations.

1.1 Applications

A method of obtaining a good approximation to the many non-linear differential equations which occur in all engineering fields would be of great utility to the engineer. Having a mathematical expression which closely approximates the solution would enable the engineer to analyze the system, and to optimize the design more rapidly, and without having to rely totally on digital computers.

CHAPTER 2

2.0 Development of the Laplace Transform for Non-Linear Functions

In order to use Laplace Transform techniques in obtaining an approximate solution to a non-linear differential equation, expressions for the transform must be derived.

2.1 The Type of Equation Defined

Consider a general non-linear differential equation of the form:

$$\left[x^{(m)}(t)\right]^k + \cdots + a_n\, x^n(t) = f(t) \qquad (2.1.1)$$

where $x^{(m)}(t)$ is the m^{th} derivative of $x(t)$; k indicates the derivative raised to the k^{th} power; and $x^n(t)$ is $x(t)$ raised to the n^{th} power.

2.2 The Method of Successive Integration by Parts

To derive an expression for the transform, use is made of successive integration by parts. Consider two time dependent functions, $x(t)$ and $y(t)$. The integral of their product is given by:

$$\int xy\, dt = x \int y\, dt - \dot{x} \iint y\, dt\, dt + \ddot{x} \iiint y\, dt\, dt\, dt - \cdots$$
$$+ (-1)^n x^{(n)} \int \left[\int y\, dt\right]^n dt \qquad (2.2.1)$$

where $\left[\int y\, dt\right]^n$ is an n-fold iterated integral. This equation can now be used to find a series expression for the Laplace Transform of $x(t)$ by letting $y(t) = e^{-st}$. Thus:

$$\int_0^\infty x e^{-st}\, dt = x \int_0^\infty e^{-st}\, dt - \dot{x} \int_0^\infty\!\!\int e^{-st}\, dt\, dt + \ddot{x} \int_0^\infty\!\!\iint e^{-st}\, dt\, dt\, dt - \cdots \qquad (2.2.2)$$

After integration the series becomes:

$$\int_0^\infty x e^{-st}\, dt = - \left.\frac{x e^{-st}}{s}\right|_0^\infty - \left.\frac{\dot{x} e^{-st}}{s^2}\right|_0^\infty - \left.\frac{\ddot{x} e^{-st}}{s^3}\right|_0^\infty - \cdots \qquad (2.2.3)$$

In evaluating the integral at the upper limit the condition must be such that:

$$\underset{t \to \infty}{\text{Limit}}\; x e^{-st} = 0 \qquad (2.2.4)$$

and similarly for all derivatives of $x(t)$. Thus, evaluating the integral at the lower limit, the result is:

$$\int_0^\infty x e^{-st}\, dt = \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \frac{\ddot{x}(0)}{s^3} + \cdots \qquad (2.2.5)$$

and since:

$$X(s) = \int_0^\infty x e^{-st}\, dt \qquad (2.2.6)$$

by definition, an infinite series expression for the transform has been derived.

2.3 Derivation of an Expression for the Transform of Derivatives

To derive an expression for the transform of a derivative of $x(t)$, the method just illustrated is used again, with $\dot{x}(t)$ in place of $x(t)$. Thus:

$$\int_0^\infty \dot{x} e^{-st}\, dt = \dot{x}\int_0 e^{-st}\, dt - \ddot{x}\int_0\!\!\int e^{-st}\, dt\, dt + \dddot{x}\int_0\!\!\int\!\!\int e^{-st}\, dt\, dt\, dt - \cdots$$

$$= \frac{\dot{x}(0)}{s} + \frac{\ddot{x}(0)}{s^2} + \frac{\dddot{x}(0)}{s^3} + \cdots \qquad (2.3.1)$$

Now, by adding and subtracting $x(0)$, and then multiplying by s/s, the result is:

$$\int_0^\infty \dot{x}^{-st}\, dt = s\left[\frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \frac{\ddot{x}(0)}{s^3} + \cdots\right] - x(0) \qquad (2.3.2)$$

The term in brackets is just X(s), so:

$$\mathcal{L}[\dot{x}(t)] = sX(s) - x(0) \tag{2.3.3}$$

a result which is well known. In a similar manner it is found that:

$$\mathcal{L}[\ddot{x}(t)] = s^2X(s) - sx(0) - \dot{x}(0) \tag{2.3.4}$$

This method can be continued and it soon becomes apparent that the expression for the transform of the m^{th} derivative is:

$$\mathcal{L}[x^{(m)}(t)] = s^mX(s) - s^{m-1}x(0) - s^{m-2}\dot{x}(0) - \cdots - sx^{(m-2)}(0) - x^{(m-1)}(0) \tag{2.3.5}$$

2.4 Transform Expression for Integral Powers of a Function

The functions considered thus far have all been linear; now the functions which cause the non-linearities will be considered. First, consider the function squared.

$$\int_0^\infty x^2e^{-st}\,dt = \int_0^\infty x(xe^{-st})dt$$

$$= x\int_0^\infty xe^{-st}\,dt - \dot{x}\iint_0^\infty x^{-st}\,dt\,dt + \ddot{x}\iiint_0^\infty x^{-st}\,dt\,dt\,dt - \cdots \tag{2.4.1}$$

Now, let

$$I_1 = x\int_0^\infty x^{-st}\,dt \tag{2.4.2}$$

The integral is the same as Equation (2.2.6), so:

$$I_1 = x(0)X(s) \tag{2.4.3}$$

Now let

$$I_2 = -\dot{x}\iint^\infty xe^{-st}\,dt\,dt \tag{2.4.4}$$

and if the inner integral is examined:

$$\int xe^{-st}\,dt = x\int e^{-st}\,dt - \dot{x}\iint e^{-st}\,dt\,dt + \ddot{x}\iiint^{-st}\,dt\,dt\,dt - \cdots$$

$$= -\frac{x^{-st}}{s} - \frac{\dot{x}^{-st}}{s^2} - \frac{\ddot{x}^{-st}}{s^3} - \cdots \tag{2.4.5}$$

This result is then put in Equation (2.4.4) to obtain:

$$I_2 = \dot{x}\left(\frac{1}{s}\int_0^\infty x^{-st}\,dt + \frac{1}{s^2}\int_0^\infty \dot{x}^{-st}\,dt + \frac{1}{s^3}\int_0^\infty \ddot{x}^{-st}\,dt + \cdots\right) \tag{2.4.6}$$

Each of the integrals can be evaluated by using the results of Section 2.3, so:

$$I_2 = \dot{x}(0)\left\{\frac{X(s)}{s} + \frac{1}{s^2}\left[sX(s) - x(0)\right] + \frac{1}{s^3}\left[s^2X(s) - sx(0) - \dot{x}(0)\right] + \cdots\right\} \tag{2.4.7}$$

If this is expanded and like terms are collected, the result is:

$$I_2 = \frac{\dot{x}(0)X(s)}{s} \tag{2.4.8}$$

Continuing in this manner it is found that:

$$\mathcal{L}[x^2(t)] = \left[x(0) + \frac{\dot{x}(0)}{s} + \frac{\ddot{x}(0)}{s^2} + \cdots\right]X(s) \tag{2.4.9}$$

But the term in brackets is just $sX(s)$, so:

$$\mathcal{L}[x^2(t)] = sX^2(s) \tag{2.4.10}$$

Thus, the transform of the square of a function has been derived. Now consider:

$$\int_0^\infty x^3e^{-st}\,dt = x\int_0^\infty x^2e^{-st}\,dt - \dot{x}\iint_0^\infty x^2e^{-st}\,dt\,dt + \cdots \tag{2.4.11}$$

The first integral has already been found, so, from Equation (2.4.10):

$$x\int_0^\infty x^2e^{-st}\,dt = x(0)sX^2(s) \tag{2.4.12}$$

The second integral is evaluated in the same manner as Equations (2.4.4) through (2.4.8). Thus:

$$-\dot{x}\iint_0^\infty x^2e^{-st}\,dt\,dt = \frac{\dot{x}(0)}{s}\,sX^2(s) \tag{2.4.13}$$

The remaining integrals are evaluated in a like manner and the outcome is:

$$\mathcal{L}[x^3(t)] = sX^2(s)\left[\frac{x(0)}{s} + \frac{\dot{x}(0)}{s} + \frac{\ddot{x}(0)}{s^2} + \cdots\right] \tag{2.4.14}$$

Once again, the term in brackets is recognized as being sX(s), so the expression can be written as:

$$\mathcal{L}[x^3(t)] = s^2X^3(s) \tag{2.4.15}$$

If higher powers of the function are examined it soon becomes apparent that:

$$\mathcal{L}[x^n(t)] = s^{n-1}X^n(s) \tag{2.4.16}$$

2.5 Transform Expression for Powers of a Derivative

Since many equations involve a derivative which is raised to a power, it will be useful to find an expression for the transform of such a function. First consider the square of the first derivative:

$$\int_0^\infty (\dot{x})^2 e^{-st}\, dt = \dot{x}\int_0^\infty \dot{x}e^{-st}\, dt - \ddot{x}\iint_0^\infty \dot{x}e^{-st} dt\, dt + \cdots \tag{2.5.1}$$

The integral in the first term has been evaluated in Section 2.3, so this term becomes:

$$\dot{x}\int_0^\infty \dot{x}e^{-st}\, dt = \dot{x}(0)[sX(s) - x(0)] \tag{2.5.2}$$

The second term is evaluated in the same manner as Equations (2.4.4) and (2.4.13). Therefore:

$$- \ddot{x}\iint_0^\infty \dot{x}e^{-st}\, dt\, dt = \frac{\ddot{x}(0)}{s}[sX(s) - x(0)] \tag{2.5.3}$$

The remaining terms in the series are evaluated in the same way, so the transform becomes:

$$\begin{aligned} \mathcal{L}\left\{[\dot{x}(t)]^2\right\} &= [sX(s) - x(0)]\left[\dot{x}(0) + \frac{\ddot{x}(0)}{s} + \frac{\dddot{x}(0)}{s^2} + \cdots\right] \\ &= [sX(s) - x(0)]\ (s)\left[\frac{\dot{x}(0)}{s} + \frac{\ddot{x}(0)}{s^2} + \cdots\right] \end{aligned} \tag{2.5.4}$$

But the second bracketed term is equivalent to sX(s) - x(0), so the transform finally becomes:

$$\mathcal{L}\left\{[\dot{x}(t)]^2\right\} = s[sX(s) - x(0)]^2 \tag{2.5.5}$$

It should be noticed that this last expression may be written as:

$$\mathcal{L}\left\{[\dot{x}(t)]^2\right\} = s\left\{\mathcal{L}[\dot{x}(t)]\right\}^2 \tag{2.5.6}$$

If the foregoing method is applied to the third power of the first derivative, the result is:

$$\mathcal{L}\left\{[\dot{x}(t)]^3\right\} = s^2\left\{\mathcal{L}[\dot{x}(t)]\right\}^3 \tag{2.5.7}$$

If higher powers of the first derivative are examined, it can be shown that:

$$\mathcal{L}\left\{[\dot{x}(t)]^n\right\} = s^{n-1}\left\{\mathcal{L}[\dot{x}(t)]\right\}^n \tag{2.5.8}$$

Now powers of the second derivative will be examined. First consider the second derivative squared:

$$\int_0^\infty (\ddot{x})^2 e^{-st}\,dt = \ddot{x}\int_0^\infty \ddot{x}e^{-st}\,dt - \dddot{x}\int_0^\infty\!\!\int \ddot{x}e^{-st}\,dt\,dt + \cdots \tag{2.5.9}$$

The integral in the first term is known to be $s^2X(s) - sx(0) - \dot{x}(0)$. The second integral is treated in the same manner as Equations (2.4.4), (2.4.13), and (2.5.3) and becomes $(1/s)[s^2X(s) - sx(0) - \dot{x}(0)]$. Thus it is found that:

$$\mathcal{L}\left\{[\ddot{x}(t)]^2\right\} = \left[\ddot{x}(0) + \frac{\dddot{x}(0)}{s} + \frac{x^{(4)}(0)}{s^2} + \cdots\right]\left[s^2X(s) - sx(0) - \dot{x}(0)\right] \tag{2.5.10}$$

The first term in brackets can be written as:

$$\begin{aligned}
s\left[\frac{\ddot{x}(0)}{s} + \frac{\dddot{x}(0)}{s^2} + \cdots\right] &= s^2\left[\frac{\ddot{x}(0)}{s} + \frac{\ddot{x}(0)}{s^2} + \frac{\dddot{x}(0)}{s^3} + \cdots\right] - s\dot{x}(0) \\
&= s^2\left[x(0) + \frac{\dot{x}(0)}{s} + \frac{\ddot{x}(0)}{s^2} + \cdots - x(0)\right] - s\dot{x}(0) \\
&= s^3\left[\frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \cdots\right] - s^2x(0) - s\dot{x}(0) \\
&= s\left\{s^2\left[\frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \cdots\right] - sx(0) - \dot{x}(0)\right\}
\end{aligned} \tag{2.5.11}$$

This is recognized as being $s[s^2X(s) - sx(0) - \dot{x}(0)]$. Thus:

$$\mathcal{L}\left\{[\ddot{x}(t)]^2\right\} = s\left\{\mathcal{L}[\ddot{x}(t)]\right\}^2 \tag{2.5.12}$$

2.6 Summary of the Transforms

If Equations (2.3.5), (2.4.16), (2.5.5), (2.5.7), and (2.5.12) are examined, it can be seen that a general expression can be written for the transform. Thus:

$$\mathcal{L}\{[x^{(m)}(t)]^n\} = s^{n-1}\{\mathcal{L}[x^{(m)}(t)]\}^n \qquad (2.6.1)$$

where m = 0, 1, 2, 3, ...; and n = 1, 2, 3, ...

CHAPTER 3

3.0 Application of the Transforms

The use of the Laplace Transform expression just derived will now be demonstrated by obtaining solutions to several equations.

3.1 Application to a Non-Linear, Homogeneous Equation

Consider a homogeneous, non-linear differential equation of the form:

$$\dot{e}(t) + Ae(t) + Be^2(t) = 0; \quad e(0) = V \tag{3.1.1}$$

Equation (2.6.1) is now used, and the transformed equation is:

$$sE(s) - e(0) + AE(s) + BsE^2(s) = 0 \tag{3.1.2}$$

This expression is now rearranged to give:

$$BsE^2(s) + (s + A)E(s) - V = 0 \tag{3.1.3}$$

This equation is quadratic in E(s), so it may be solved for E(s) through the use of the familiar quadratic formula. Thus:

$$E(s) = \frac{-(s + A) \pm \sqrt{(s + A)^2 + 4BVs}}{2Bs} \tag{3.1.4}$$

Now this result is rewritten to put it into a more convenient and useful format. Thus:

$$E(s) = \frac{(s+A)}{2Bs}\left[-1 \pm \left(1 + \frac{4BVs}{(s+A)^2}\right)^{\frac{1}{2}}\right] \tag{3.1.5}$$

Now, if the positive radical is used and is expanded in a binomial series, the resulting expression for E(s) is:

$$E(s) = \frac{V}{s+A} - \frac{BV^2s}{(s+A)^3} + \frac{2B^2V^3s^2}{(s+A)^5} - \frac{5B^3V^4s^3}{(s+A)^7} + \cdots \tag{3.1.6}$$

Thus, an infinite series expression has been obtained for E(s). Now the inverse transform is to be found for each term of the series [ref. 3],

and the resulting expression is:

$$e(t) \cong Ve^{-At}\left[1 - BVt\left(1 - \frac{At}{2}\right) + (BVt)^2\left(1 - \frac{2At}{3} + \frac{A^2t^2}{12}\right)\right.$$
$$\left. - (BVt)^3\left(\frac{5}{6} - \frac{5At}{8} + \frac{A^2t^2}{8} - \frac{A^3t^3}{144}\right) + \cdots\right] \quad (3.1.7)$$

An infinite series approximation has been found for e(t). However, for the original equation, an exact solution is known [ref. 4]:

$$e(t) = Ve^{-At}\left[1 - \frac{BV}{A}\left(e^{-At} - 1\right)\right]^{-1} \quad (3.1.8)$$

By expanding the term in brackets, the two solutions may be compared. Thus, after using a binomial expansion, the exact solution is:

$$e(t) = Ve^{-At}\left[1 + \frac{BV}{A}\left(e^{-At} - 1\right) - \frac{B^2V^2}{A^2}\left(e^{-At} - 1\right)^2\right.$$
$$\left. + \frac{B^3V^3}{A^3}\left(e^{-At} - 1\right)^3 - \cdots\right] \quad (3.1.9)$$

Now, if each of the expotentials is expanded in a series, the result is:

$$e(t) = Ve^{-At}\left[1 - BVt\left(1 - \frac{At}{2} + \frac{A^2t^2}{6} - \frac{A^3t^3}{24} + \cdots\right)\right.$$
$$\left. + B^2V^2t^2\left(1 - At + \frac{7A^2t^2}{12} - \frac{A^3t^3}{4} + \cdots\right) - \cdots\right]$$
$$(3.1.10)$$

Notice the similarity between this equation and the approximate solution in Equation (3.1.7). Each sub-series in the approximate solution is truncated, since the second term of the s-domain expression yields two time-domain terms; the third term of the s-domain expression gives three time-domain terms; and so on.

From an analysis of the parameters A, B, and V, it is apparent that $BV < A$, and further, the smaller BV is with respect to A the better the approximation will be. Table 1 shows a comparison of the exact and approximate solutions for two sets of B, V, and A. Figure 1

shows a graph of the exact and approximate solutions when A = 1.0, B = 0.125, and V = 1.0.

TABLE 1

A. A = 1.0, B = 0.5, V = 1.0

Time	Exact	Approximate	Error
0.0	1.000	1.000	0.000
0.5	0.507	0.473	0.034
1.0	0.279	0.257	0.022
1.5	0.161	0.174	-0.013
2.0	0.094	0.118	-0.024
2.5	0.056	0.057	-0.001
3.0	0.034	-0.010	0.044

B. A = 1.0, B = 0.125, V = 1.0

Time	Exact	Approximate	Error
0.0	1.000	1.000	0.000
0.5	0.578	0.577	0.001
1.0	0.341	0.344	-0.003
1.5	0.203	0.214	-0.011
2.0	0.122	0.139	-0.017
2.5	0.074	0.095	-0.021
3.0	0.044	0.067	-0.023
3.5	0.027	0.048	-0.021
4.0	0.016	0.035	-0.019
4.5	0.010	0.026	-0.016
5.0	0.006	0.019	-0.013

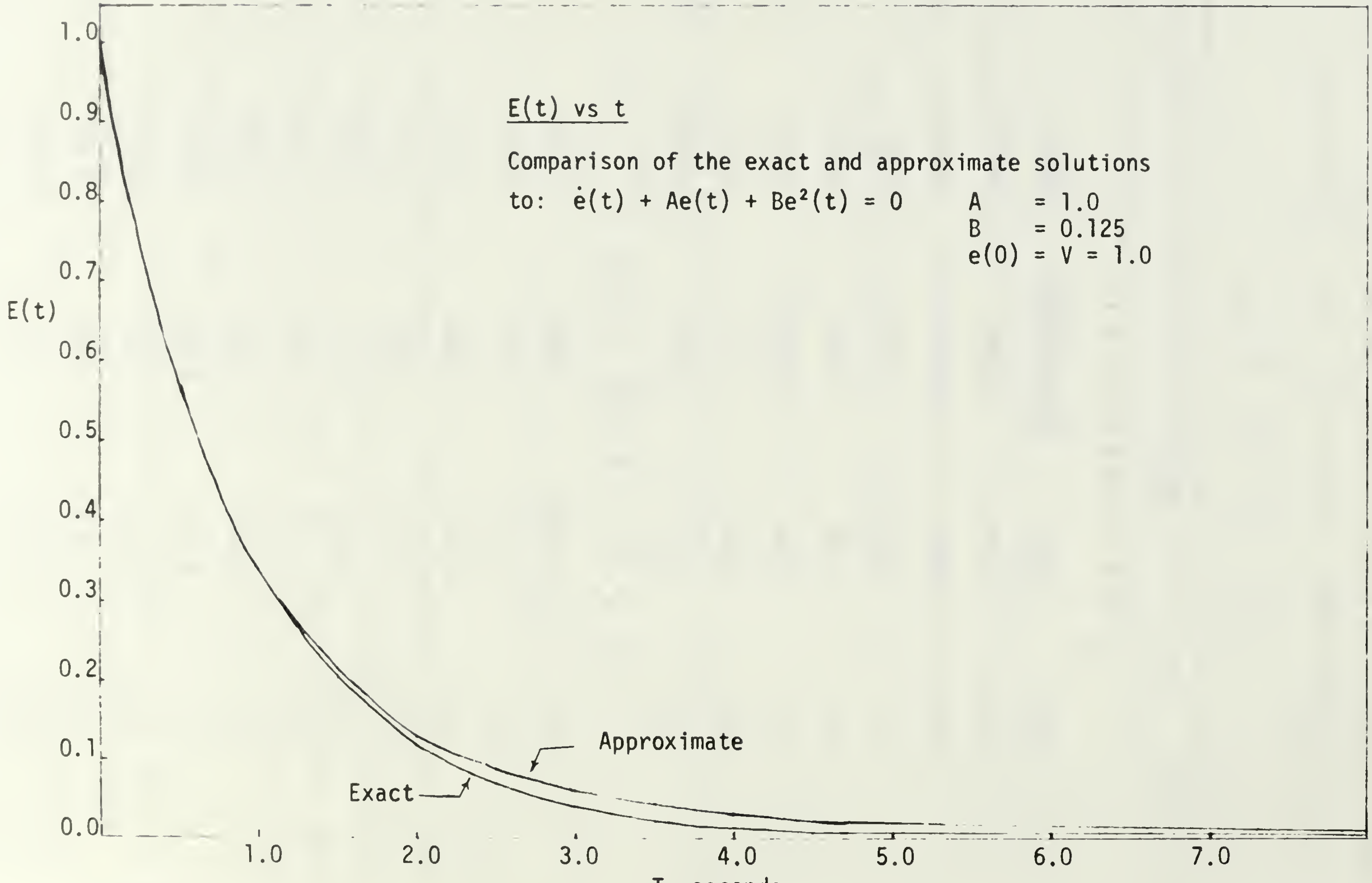

E(t) vs t
Comparison of the exact and approximate solutions
to: $\dot{e}(t) + Ae(t) + Be^2(t) = 0$
A = 1.0
B = 0.125
e(0) = V = 1.0
E(t)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
T, seconds
Approximate
Exact

Figure 1

3.2 Application to a Non-Linear, Non-Homogeneous Equation

Consider an equation of the form:

$$M\dot{v} + kv^2 = Mg \tag{3.2.1}$$

This is the equation of motion of a falling object with the retarding effect of air resistance considered. It is assumed that the retarding force is proportional to the velocity squared. If the equation is written as:

$$\dot{v} + hv^2 - g = 0 \tag{3.2.2}$$

where

$$v(0) = 0 \tag{3.2.3}$$

and

$$h = k/M, \tag{3.2.4}$$

then an approximate solution to the equation can be found by applying Equation (2.6.1) as before. Thus:

$$sV(s) - v(0) + hsV^2(s) - \frac{g}{s} = 0 \tag{3.2.5}$$

This expression is rearranged to get:

$$hs^2V^2(s) + s^2V(s) - g = 0 \tag{3.2.6}$$

Once again the result is a quadratic equation in V(s), and it is solved for V(s) by using the quadratic formula. Thus:

$$V(s) = \frac{1}{2hs^2}\left[- s^2 \pm \sqrt{s^4 + 4ghs^2}\right] \tag{3.2.7}$$

This equation is now rearranged to get the same form as that used in Equation (3.1.5):

$$V(s) = \frac{1}{2h}\left[- 1 \pm \left(1 + \frac{4gh}{s^2}\right)^{\frac{1}{2}}\right] \tag{3.2.8}$$

If the term in parentheses is expanded in a binomial series and the positive sign in front of the radical is used, the result, after

simplification, is:

$$V(s) = \frac{g}{s^2} - \frac{g^2h}{s^4} + \frac{2g^3h^2}{s^6} - \frac{5g^4h^3}{s^8} + \frac{14g^5h^4}{s^{10}} - \frac{42g^6h^5}{s^{12}} + \cdots \quad (3.2.9)$$

and the time-domain solution is obtained by finding the inverse transform of each term of the series. Thus:

$$v(t) \cong gt - \frac{g^2ht^3}{3!} + \frac{2g^3h^2t^5}{5!} - \frac{5g^4h^3t^7}{7!} + \frac{14g^5h^4t^9}{9!} - \cdots \quad (3.2.10)$$

So an approximate solution has been found. The exact solution is known [ref. 5]:

$$v(t) = \left(\frac{g}{h}\right)^{\frac{1}{2}}\left[\frac{1 - e^{-2(gh)^{\frac{1}{2}}t}}{1 + e^{-2(gh)^{\frac{1}{2}}t}}\right] \quad (3.2.11)$$

where $(g/h)^{\frac{1}{2}}$ is the terminal velocity of the falling object. If the numerator and denominator of this expression are multiplied by $e^{(gh)^{\frac{1}{2}}t}$, then:

$$v(t) = \left(\frac{g}{h}\right)^{\frac{1}{2}} \tanh\,(gh)^{\frac{1}{2}}t \quad (3.2.12)$$

The hyperbolic tangent can be written as an infinite series, and in this form the exact solution is:

$$v(t) = \left(\frac{g}{h}\right)^{\frac{1}{2}}\left[(gh)^{\frac{1}{2}} - \frac{(gh)^{\frac{3}{2}}t^3}{3} \quad \frac{2(gh)^{\frac{5}{2}}t^5}{15} - \frac{17(gh)^{\frac{7}{2}}t^7}{315} + \cdots\right] \quad (3.2.13)$$

To compare the two solutions, Equation (3.2.10) is rewritten as:

$$v(t) \cong \left(\frac{g}{h}\right)^{\frac{1}{2}}\left[(gh)^{\frac{1}{2}} - \left(\frac{1}{2}\right)\frac{(gh)^{\frac{3}{2}}t^3}{3} + \left(\frac{1}{8}\right)\frac{2(gh)^{\frac{5}{2}}t^5}{15} - \left(\frac{5}{272}\right)\frac{17(gh)^{\frac{7}{2}}t^7}{315} + \cdots\right] \quad (3.2.14)$$

The series are similar in form, but each term of the approximate solution is smaller than the corresponding term in the exact solution. In order to get a numerical comparison, consider the falling object to be a spherical rock with density (ρ) of 2600 kilograms/meter^3. The following parameters must be defined and evaluated [ref. 6]:

$$M = \rho V$$

$$k = 0.5\ \rho_a\ C_d\ A$$

where M is the mass of the object, V is its volume, and A is its surface area. The other constants are:

ρ_a = 1.293 kilograms/meter^3 (Actually, ρ_a is a function of air pressure and temperature, but it will be assumed to be very nearly constant for this analysis.)

C_d = 0.5, the drag coefficient for a sphere

$A = \pi D^2$, the surface area of a sphere

$V = \frac{\pi D^3}{6}$, the volume of a sphere

g = 9.8 meters/second^2, the acceleration due to gravity

Thus, if the sphere has a diameter of 0.1 meter, the exact solution is:

$$v(t) = 36.4 \tanh 0.27t \tag{3.2.15}$$

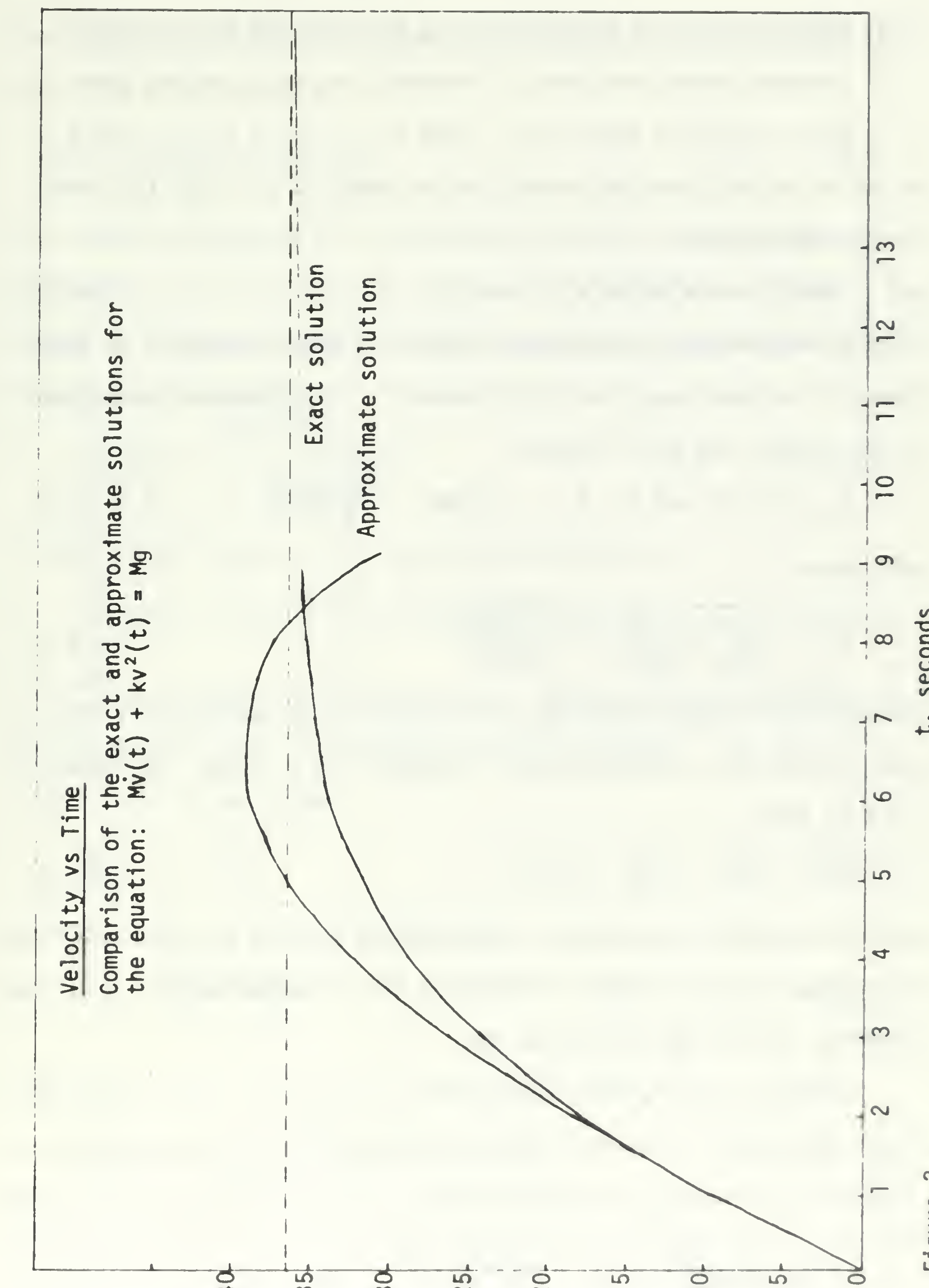
and the approximate solution becomes

$$v(t) \cong 36.4\left[0.27t - \frac{(0.27t)^3}{6} + \frac{(0.27t)^5}{60} - \frac{(0.27t)^7}{1008} + \frac{(0.27t)^9}{25920} - \cdots\right] \tag{3.2.16}$$

The comparison of these two expressions is shown in Table II and a graph in Figure 2. In the graph, notice that the exact solution asymptotically approaches the terminal velocity 36.4 meters/second, while the approximate solution overshoots that value, and after about seven seconds decreases monotonically. This is due to the fact that the highest power term in the series was preceded by a negative sign, and as t became large this term overpowered the others. The first term omitted after the series was truncated was positive, so if it were included, the graph would end by increasing monotonically. The hump is caused by the lower degree terms being predominant when t is small.

TABLE II

Time	Exact	Approximate	Error
0.0	0.000	0.000	0.000
0.5	4.884	4.885	-0.001
1.0	9.596	9.682	-0.086
1.5	11.839	12.021	-0.182
2.0	17.945	18.679	-0.734
2.5	21.413	22.731	-1.318
3.0	24.373	26.403	-2.030
3.5	26.845	29.649	-2.804
4.0	28.872	32.440	-3.568
4.5	30.510	34.757	-4.247
5.0	31.816	36.598	-4.782
5.5	32.847	37.968	-5.121
6.0	33.656	38.879	-5.223
6.5	34.287	39.337	-5.050
7.0	34.776	39.328	-4.552
7.5	35.153	38.788	-3.635
8.0	35.444	37.574	-2.130
8.5	35.668	35.399	0.269
9.0	35.840	31.750	4.090
9.5	35.972	25.768	10.204
10.0	36.073	16.077	19.996



Velocity vs Time
Comparison of the exact and approximate solutions for the equation: $M\dot{v}(t) + kv^2(t) = Mg$
Exact solution
Approximate solution
v(t)
0
5
10
15
20
25
30
35
40
t, seconds
1
2
3
4
5
6
7
8
9
10
11
12
13

Figure 2

3.3 Application to an Equation with a Third Degree Non-Linearity

Consider now a non-linear differential equation of the form:

$$\ddot{x}(t) + \omega_o^2 x(t) + hx^3(t) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0 \qquad (3.3.1)$$

After using the transform expression of Equation (2.6.1), the transformed equation is:

$$hs^2X^3(s) + (s^2+\omega_o^2)X(s) - As = 0 \qquad (3.3.2)$$

This is in the so-called "normal" form of a cubic equation, in that there is no term involving $X^2(s)$ present. Thus there are solutions $X_1(s)$, $X_2(s)$, and $X_3(s)$; where:

$$X_1 = A + B, \text{ and } X_2, X_3 = -\tfrac{1}{2}(A+B) \pm \frac{i\sqrt{3}}{2}(A-B) \qquad (3.3.3)$$

and where:

$$A,B = \sqrt[3]{\frac{A}{2hs} \pm \sqrt{\frac{A^2}{4h^2s^2} + \frac{(s^2+\omega_o^2)^3}{27h^3s^6}}} \qquad (3.3.4)$$

It can be seen that expanding this expression and determining which root to use is a formidable task. However, it is known from Equation (2.2.5) that:

$$X(s) = \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \frac{\ddot{x}(0)}{s^3} + \cdots \qquad (3.3.5)$$

and if the initial conditions from Equation (3.3.1) are used and if it is assumed that all higher derivatives can be neglected at t = 0, then Equation (3.3.2) may be written as:

$$hAsX^2(s) + (s^2+\omega_o^2)X(s) - As = 0 \qquad (3.3.6)$$

Thus, the degree of the equation has been reduced by one and the solution can be obtained in the same manner as in the previous two sections:

$$X(s) = \frac{(s^2+\omega_o^2)}{2hAs}\left[-1 \pm \left(1 + \frac{4hA^2s^2}{(s^2+\omega_o^2)^2}\right)^{\frac{1}{2}}\right] \qquad (3.3.7)$$

Then, expanding the term in brackets as in previous solutions, the result is:

$$X(s) = \frac{As}{s^2+\omega_o^2} - \frac{hA^3s^3}{(s^2+\omega_o^2)^3} + \frac{2h^2A^5s^5}{(s^2+\omega_o^2)^5} + \cdots \qquad (3.3.8)$$

Now the inverse transform is found for each term of the series. For the first term, then:

$$\mathcal{L}^{-1}\left[\frac{As}{s^2+\omega_o^2}\right] = A \cos \omega_o t \tag{3.3.9}$$

The inverse transforms for the succeeding terms are more difficult to evaluate, as no expression of the form $s^n/(s^2+\omega_o^2)^n$ can be found in the tables of Laplace Transforms available. However, one handbook [ref. 7] does have the transform:

$$\frac{s}{(s^2+\omega_o^2)^n} = \frac{\pi^{\frac{1}{2}} t^{n-\frac{1}{2}} J_{n-\frac{3}{2}}(\omega_o t)}{2^{n-\frac{1}{2}}\Gamma(n)\omega_o^{n-\frac{3}{2}}} \tag{3.3.10}$$

Further, the table [ref. 8] has the relation that:

$$s^n g(s) = f^{(n)}(t) + \sum_{k=o}^{n-1} f^{(k)}(0)s^{n-1-k}, \; n = 1, 2, 3, \cdots \tag{3.3.11}$$

if $f^{(k)}(0) = 0$ for $k = 0, 1, \ldots, n-1$. Thus, these two relations can be combined to yield the desired inverse transforms. So, for the second term of the series:

$$g(s) = \frac{s}{(s^2+\omega_o^2)^3} \tag{3.3.12}$$

and its inverse transform is given by:

$$f(t) = \frac{\pi^{\frac{1}{2}} t^{\frac{5}{2}} J_{\frac{3}{2}}(\omega_o t)}{2^{\frac{5}{2}}\Gamma(3)\omega_o^{\frac{3}{2}}} \tag{3.3.13}$$

Then, the Bessel Function is given by [ref. 9]:

$$J_{\frac{3}{2}}(\omega_o t) = \left(\frac{2}{\pi\omega_o t}\right)^{\frac{1}{2}}\left(\frac{\sin \omega_o t}{\omega_o t} - \cos \omega_o t\right) \tag{3.3.14}$$

Upon substituting this expression into Equation (3.3.13), the resultant inverse transform of the second term is, after applying Equation (3.3.11):

$$f(t) = \frac{hA^3}{8\omega_o^2}\left(3\,\omega_o t \sin \omega_o t + \omega_o^2 t^2 \cos \omega_o t\right) \tag{3.3.15}$$

This method is then applied to the other terms of the series and the resulting expression is:

$$x(t) \cong A \cos y - \frac{hA^3}{8\omega_o^2}(3y \sin y + y^2 \cos y) + \frac{h^2A^5}{\omega_o^4}\left(\frac{5}{64} y \sin y - \frac{5y^2}{64} \cos y + \frac{5y^3}{96} \sin y + \frac{y^4}{192} \cos y\right) - \cdots \tag{3.3.16}$$

where, for convenience, $y = \omega_o t$. Another approximate solution is given by Cunningham [ref. 10], using the Ritz method. The result is:

$$x(t) \cong A \cos\sqrt{\omega_o^2 + 0.75hA^2}\ t \tag{3.3.17}$$

Using the arbitrary values $\omega_o = 1$, $h = 1$, and $A = 1$, a comparison of these two solutions is shown in Table III and a graph in Figure 3. With the arbitrary values selected, the new method yields good results, when compared to the Ritz method, up to about one second. Examination of the series shows that better results could be obtained over a longer period of time if the quantity A were made less than unity.

In the method originally proposed by Baycura [ref. 1], the transformed equation was the same as that which would result if the relation X(s) equals A/s were used again in Equation (3.3.2):

$$X(s) = \frac{s}{s^2 + (1 + 1)} \tag{3.3.18}$$

For this expression, then the inverse transform is:

$$x(t) \cong \cos\sqrt{2}\ t \tag{3.3.19}$$

assuming that $A = h = \omega_o = 1$. At $t = 2$, $x(t) = -0.951$, and at $t = 1$, $x(t) = 0.156$. Thus, if desired, the equation in s can be reduced to a linear expression, and this will yield a solution which is not in the form of an infinite series.

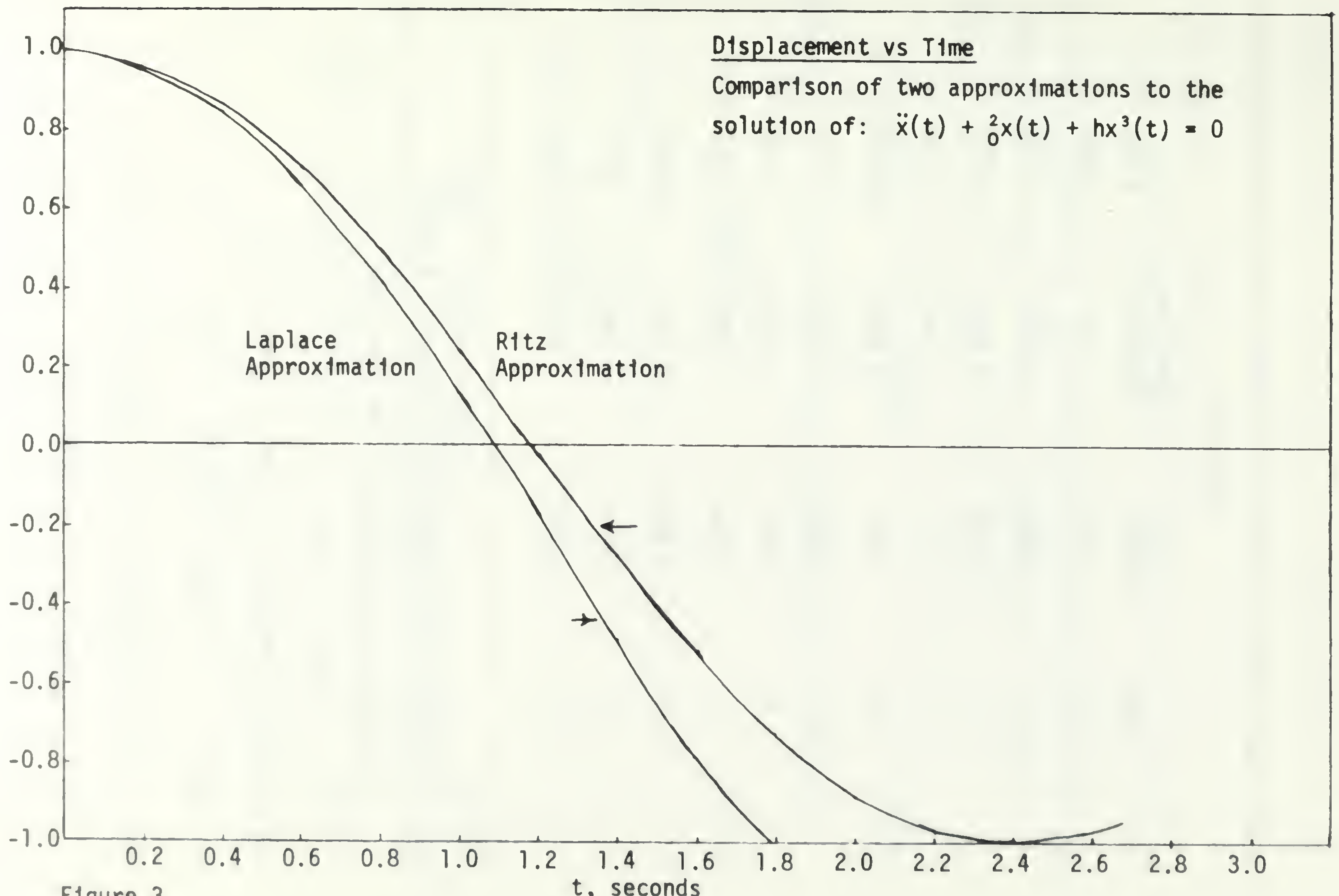

Displacement vs Time
Comparison of two approximations to the
solution of: $\ddot{x}(t) + {}^2_0x(t) + hx^3(t) = 0$
Laplace
Approximation
Ritz
Approximation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
t, seconds

Figure 3

TABLE III

Time	Ritz Method	Laplace Method	Error
0.0	1.000	1.000	0.000
0.2	0.965	0.960	0.005
0.4	0.863	0.844	0.019
0.6	0.701	0.658	0.043
0.8	0.490	0.416	0.074
1.0	0.245	0.134	0.111
1.2	-0.017	-0.173	0.156
1.4	-0.278	-0.485	0.207
1.6	-0.519	-0.786	0.267
1.8	-0.724	-1.060	0.336
2.0	-0.880	-1.290	0.410

3.4 An Equation with a Derivative Raised to a Power

Consider now an equation of the form [ref. 11]:

$$\ddot{x}(t) + [\dot{x}(t)]^2 + x(t) + k = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0 \qquad (3.4.1)$$

The transformed equation is found as before, thus:

$$s^3X^2(s) - (s^2-1)X(s) + \frac{k}{s} = 0 \qquad (3.4.2)$$

This is then solved for X(s) as before, and the result is:

$$X(s) = \frac{(s^2-1)}{2s^3}\left[1 \pm \left(1 - \frac{4ks^3}{(s^2-1)^2}\right)^{\frac{1}{2}}\right] \qquad (3.4.3)$$

The term in brackets is then expanded as a binomial series, but this time the negative sign in front of the radical is chosen to avoid a delta function in the solution, so that:

$$X(s) = \frac{k}{s^2-1} + \frac{k^2s^3}{(s^2-1)^3} + \frac{2k^3s^6}{(s^2-1)^5} + \cdots \qquad (3.4.4)$$

The inverse transform of the first term is k sinh t, but for the succeeding terms the same difficulty is encountered as in Section 3.3. However, the table of transforms [ref. 12] lists:

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2-1)^n}\right] = \frac{\pi^{\frac{1}{2}}t^{n-\frac{1}{2}}\, I_{n-\frac{3}{2}}(t)}{2^{n-\frac{1}{2}}\Gamma(n)} \qquad (3.4.5)$$

where $I_{n-\frac{3}{2}}(t)$ is the modified Bessel function [ref. 13] given by:

$$I_N(t) = \sum_{k=o}^{\infty} \frac{(t/2)^{N+2k}}{k!\Gamma(N+k+1)} \qquad (3.4.6)$$

For the second term of the series, n = 3, N = 3/2, and:

$$I_{\frac{3}{2}}(t) = \sum_{k=o}^{\infty} \frac{(t/2)^{\frac{3}{2}+2k}}{k!\Gamma(\frac{5}{2}+k)}$$

$$= \frac{(t/2)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} + \frac{(t/2)^{\frac{7}{2}}}{\Gamma(\frac{7}{2})} + \frac{(t/2)^{\frac{11}{2}}}{\Gamma(\frac{9}{2})} + \cdots \qquad (3.4.7)$$

and to evaluate the Gamma function [ref. 14]:

$$\Gamma(m+\tfrac{1}{2}) = \frac{1.3.5.7...(2m-1)}{2^m}\sqrt{\pi} \qquad (3.4.8)$$

Thus:

$$I_{\frac{3}{2}}(t) = \frac{4(t/2)^{\frac{3}{2}}}{3\sqrt{\pi}} + \frac{8(t/2)^{\frac{7}{2}}}{15\sqrt{\pi}} + \frac{16(t/2)^{\frac{11}{2}}}{105\sqrt{\pi}} + \cdots \qquad (3.4.9)$$

This result is substituted in Equation (3.4.5) to get:

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2-1)^3}\right] = \frac{1}{32}\left[\frac{4t^4}{3} + \frac{t^6}{15} + \frac{t^8}{105} + \cdots\right] \qquad (3.4.10)$$

Examination of this expression shows that the system is unstable, since the terms are always positive and the series has no limit, increasing without bound as t becomes large.

CHAPTER 4

4.0 Conclusions

It has been established that a method for obtaining an approximate solution to a non-linear differential equation has been derived, and that the method uses techniques and characteristics inherent to the Laplace Transform. Further, it has been shown that sufficient accuracy exists in the approximation to make it useful in many engineering applications.

As in any approximation which uses a truncated infinite series to obtain a numerical answer to a problem, extreme care must be exercised to ensure that the approximation is accurate within a reasonable degree. This means that the parameters must be examined closely to avoid exceeding the capability of the approximation. It also means that the approximation may indeed only be accurate over a small part of the entire solution.

The results included here are the outcome of what really amounts to preliminary investigation only. One question that remains is that, since the solutions obtained in Sections 3.1 and 3.2 are so similar in form to the exact solutions, and since the solution seems exact in the Laplace expression, why isn't it, in fact, exact? After all, one major attraction of the Laplace Transform in solving linear equations is that the solution may be obtained by manipulating the transformed equation algebraically, and that is how the solutions were arrived at here. Thus, it seems that further research must be done in the area of this non-linear transform.

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13. ABSTRACT

At the present time the primary method of obtaining solutions to non-linear differential equations is by means of the digital computer and numerical techniques. A method is here proposed to find an approximate mathematical expression through the use of Laplace Transform techniques. Thus, the Laplace Transform concept is extended to the solution of non-linear differential equations.

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